THE EXISTENCE OF NEGATIVE MOMENTS OF CONTINUOUS DISTRIBUTIONS

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Abstract

The question of existence of negative moments of a continuous probability density function is explored. Sufficient conditions for both existence and nonexistence of negative moments are given. The conditions are easy to check and, although they do not constitute a necessary and sufficient condition for existence, they will usually decide the case for most commonly encountered density functions. For positive densities, a necessary and sufficient condition for existence (which is much harder to verify) is also presented.

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1. Introduction. In an introductory (post-calculus) probability and statistics course, students spend much time in evaluation of expectations of random variables. For the most part, such evaluations are confined to positive moments of well-behaved distributions and, hence, the question of existence of these moments is rarely an issue. With the possible exception of the Cauchy distribution, the existence of at least two positive moments is usually a foregone conclusion. This is not the case, however, if one is attempting the evaluation of negative moments, for here nonexistence is a much more common occurrence.

In many practical applications one is led quite naturally to the evaluation of negative moments of a random variable. We cite three simple examples.

EXAMPLE 1.1. The exponential distribution with parameter λ, given by

$$f(x) = \lambda e^{-\lambda x}, \qquad 0 \le x < \infty \quad , \tag{1.1}$$

is often used as a model for life-testing situations. If one observation is taken from this distribution, the standard method of maximum likelihood estimation would lead to using $1/x$ as an estimate of λ. Does the expected value, $E(1/x)$, exist?

EXAMPLE 1.2. In simple linear regression, one assumes the model $y = \alpha + \beta x$. Usually, interest is in predicting y from x using a fitted equation $y = \hat{\alpha} + \hat{\beta} x$ where, under standard assumptions, $\hat{\alpha}$ and $\hat{\beta}$ are normally distributed. Sometimes, however, the interest is in inverse regression; predicting x from y. The

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natural estimate of x, for a given y, is then $(y - \hat{\alpha})/\hat{\beta}$. Does $E[(y - \hat{\alpha})/\hat{\beta}]$ exist?

EXAMPLE 1.3. If a sample is being drawn from a population with mean μ and standard deviation σ, a measure of the amount of variation in the population (which is particularly popular in the engineering sciences) is the coefficient of variation $\sigma/|\mu|$. A natural estimate of this quantity is $s/|\bar{x}|$, where $\bar{x}$ and s are the sample mean and sample standard deviation, respectively. Does $E(s/|\bar{x}|)$ exist?

In general, the theory behind the existence of negative moments is quite difficult, and not nearly as complete as that involving positive moments. There, one is led into the (quite elegant) theory of characteristic functions, the Fourier transforms of the density function. However, by restricting attention to distributions with continuous density functions, the tools available to the student in an introductory course are sufficient to establish some fairly strong results; certainly strong enough to deal with most continuous distributions that are likely to be encountered.

In Section 2 we present two theorems, one giving a sufficient condition for existence of the first negative moment, and the other condition giving a sufficient condition for nonexistence. Both conditions are easy to check; they do not involve evaluating an integral. Although these theorems are strong enough to handle most common distributions, they do not provide a necessary and sufficient condition for the existence of the first negative moment. In Section 3 we present a distribution which is not covered by these theorems, and prove a theorem which gives a necessary and sufficient condition for the existence of the first negative moment of a positive random variable. Section 4 discusses some immediate generalizations to higher order negative moments.

2. Conditions for Existence and Nonexistence. The difficulties experienced in calculating expressions such as $E(X^{-1})$ will occur near $X=0$. For example, if X has a discrete probability density function (pdf) with positive mass at $X=0$, $E(X^{-1})$ will be infinite. Chao and Strawderman [1] sidestep this problem by adding a value to the random variable which makes it positive with probability one. We will only be concerned here with the continuous case, which, in some respects, is more complex.

Let X be a random variable with continuous pdf, $f(x)$. For any $\gamma,\ \delta>0$ we can write

$$E(X^{-1}) = \int_{-\infty}^{\infty} x^{-1} f(x)dx = \left[\int_{-\infty}^{-\gamma} + \int_{-\gamma}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty}\right][x^{-1} f(x)dx] \quad . \tag{2.1}$$

Clearly, both $\int_{-\infty}^{-\gamma} x^{-1} f(x)dx$ and $\int_{\delta}^{\infty} x^{-1} f(x)dx$ are finite (since x^{-1} is bounded over these regions), hence $E(X^{-1})<\infty$ if and only if

$$\int_{-\gamma}^{0} x^{-1} f(x)dx<\infty \qquad \text{and} \qquad \int_{0}^{\delta} x^{-1} f(x)dx<\infty \tag{2.2}$$

for some $\gamma,\ \delta>0$. Note that $E(X^{-1})<\infty$ if and only if $E(|X|^{-1})<\infty$. We have the following simple condition for nonexistence of $E(|X|^{-1})$.

THEOREM 2.1. Let $f(x)$ be a continuous pdf on (a,b), $-\infty\le a<b\le\infty$ ($f(x)=0$ on $(-\infty,a]\cup[b,\infty)$). If one of the following conditions is true:

$$\text{(i)} \quad a<0<b \quad \text{and} \quad f(0)>0 \ ,$$

$$\text{(ii)} \quad a=0 \quad \text{and} \quad \lim_{x\to 0^+} f(x)>0 \ ,$$

$$\text{(iii)} \quad b=0 \quad \text{and} \quad \lim_{x\to 0^-} f(x)>0 \ ,$$

then $E(|X|^{-1})=\infty$.

Proof. If (i) is true, from the continuity of $f(x)$ there exists ϵ, $\delta > 0$ such that $f(x) > \epsilon$ when $x \in (-\delta, \delta)$, and δ can be chosen such that $(-\delta, \delta) \subset (a,b)$. Then, using (2.1), we have

$$E(|X|^{-1}) \geq \int_{-\delta}^{0} |x|^{-1} f(x)dx + \int_{0}^{\delta} |x|^{-1} f(x)dx \geq 2\epsilon \int_{0}^{\delta} x^{-1}\, dx = \infty \quad . \tag{2.3}$$

If (ii) is true, there exists ϵ, $\delta > 0$ such that $f(x) > \epsilon$ when $x \in (0, \delta)$, and δ can be chosen to satisfy $0 < \delta < b$. Then

$$E(|X|^{-1}) \geq \int_{0}^{\delta} |x|^{-1} f(x)dx \geq \epsilon \int_{0}^{\delta} x^{-1}\, dx = \infty \quad . \tag{2.4}$$

The proof is similar for condition (iii).

This theorem shows, in particular, that a continuous pdf on $(-\infty, \infty)$ will not, in general, have a first negative moment. (One could, of course, construct a distribution continuous on $(-\infty, \infty)$, with $f(0) = 0$, such that the first negative moment does exist, but this would be a rather uncommon distribution.) The far more interesting case, to which we will restrict attention, is that of positive random variables.

If $f(x)$ is a continuous pdf on $(0, \infty)$ it is natural to inquire, in light of Theorem 2.1, if $\lim_{x \to 0^+} f(x) = 0$ is a sufficient condition for $E(X^{-1}) < \infty$. This is not the case, for it is the rate at which $f(x)$ approaches zero that is important.

THEOREM 2.2. <u>If</u> $f(x)$ <u>is a continuous pdf on</u> $(0, \infty)$, <u>and there exists</u> $\alpha \geq 0$ <u>such that</u>

$$\lim_{x \to 0} [f(x)/x^{\alpha}] < \infty \quad , \tag{2.5}$$

<u>then</u> $E(X^{-1}) < \infty$.

Proof. If $\alpha \geq 0$ satisfies (2.5), then there exists a finite constant M and $\delta > 0$ such that $|f(x)/x^{\alpha}| \leq M$ when $0 \leq x \leq \delta$. Hence,

$$\int_0^\delta x^{-1} f(x)dx = \int_0^\delta x^{\alpha-1}[f(x)/x^\alpha]dx \le M\int_0^\delta x^{\alpha-1}dx$$
$$= \frac{M}{\alpha}\delta^\alpha < \infty \quad , \tag{2.6}$$

so, using (2.2), $E(X^{-1})$ exists.

Theorem 2.2 has an immediate corollary for differentiable density functions.

COROLLARY 2.1. Let $f(x)$ be a continuous pdf on $(0,\infty)$, with $f(0)=0$. If $f'(0)$ exists and is finite, then $E(X^{-1})<\infty$.

Proof. From the definition of a derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x} \quad , \tag{2.7}$$

since $f(0)=0$. The existence of $E(X^{-1})$ now follows from Theorem 2.2 with $\alpha=1$.

As mentioned before, the results of this section are strong enough to determine the existence or nonexistence of first negative moments for many common distributions. We illustrate this with some examples.

EXAMPLE 2.1. Let $f(x)$ be the uniform density on (a,b), i.e.,

$$\begin{aligned} f(x) &= \frac{1}{b-a} \qquad \text{if} \qquad a<x<b \quad , \\ &= 0 \qquad \text{otherwise} \quad . \end{aligned} \tag{2.8}$$

If $a<0<b$, then $f(0)=(b-a)^{-1}>0$ and hence, from Theorem 2.1, $E(|X|^{-1})=\infty$. If either $a=0$ or $b=0$, it again follows from Theorem 2.1 that $E(|X|^{-1})=\infty$. If $[a,b]$ does not contain zero, then $E(|X|^{-1})<\infty$.

EXAMPLE 2.2. If $f(x)$ is the normal density with mean μ and standard deviation σ,

$$f(x) = \frac{1}{(2\pi\sigma)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}, \qquad -\infty<x<\infty \quad , \tag{2.9}$$

then $f(0) > 0$ and $E(X^{-1})$ does not exist. This example covers the situation outlined in Examples 1.2 and 1.3 (if the population is assumed to be normal), and shows that neither of the 'natural' estimates presented there have finite expectation.

EXAMPLE 2.3. The gamma distribution with parameters $r > 0$ and $\lambda > 0$ has pdf

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \qquad 0 \le x < \infty \quad , \tag{2.10}$$

where $\Gamma(\cdot)$ denotes the gamma function. (The exponential distribution of Example 1.1 is a special case, having $r = 1$.) It is easy to check that $f(0) > 0$ if $r \le 1$, showing the nonexistence of $E(X^{-1})$ if $r \le 1$. If $r > 1$, then

$$\lim_{x \to 0^+} \frac{f(x)}{x} = 0 \quad , \tag{2.11}$$

showing that $E(X^{-1}) < \infty$ if and only if $r > 1$.

Although Theorems 2.1 and 2.2 do not provide one necessary and sufficient condition for the existence of the first negative moment, the above examples show that they are strong enough to decide the case for many common distributions. Perhaps their main virtue lies in the simplicity of their conditions; one need only evaluate the density function to see if the theorem applies. In the next section, we shall investigate a density for which these theorems fail to give an answer, and prove a necessary and sufficient condition for existence of the first negative moment. Unfortunately, the simplicity in checking the condition is forfeit, for we no longer can state it in terms of the density function.

3. A Necessary and Sufficient Condition. The establishment of necessary and sufficient conditions for the existence of moments is, in general, a difficult endeavor. Indeed, when dealing with positive moments, one is quickly led into the theory of characteristic functions (see, e.g., [3] or [4]). The theory of the existence of negative moments cannot be tied in with that of characteristic functions, so, in a sense, is less elegant. However, for the case of continuous, positive density functions, we can present a relatively simple necessary and sufficient condition. This condition, unfortunately, is not nearly as easy to verify as those in Section 2, for it is in terms of an integral rather than a limit. As far as we know, one cannot express a necessary and sufficient condition in terms of a limit.

We begin with an example of a density which is not covered by the theorems of Section 2.

EXAMPLE 3.1. For any constant a, $0 < a < 1$, define the density $f_a(x)$ by

$$
\begin{aligned}
f_a(x) &= [\log(1/x)]^{-1} \Big/ \int_0^a [\log(1/t)]^{-1}\, dt \qquad 0 < x < a \quad , \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \quad .
\end{aligned}
\tag{3.1}
$$

It is easy to check that, for every a, $0 < a < 1$,

$$
\begin{aligned}
&\lim_{x\to 0} f_a(x) = 0 \quad , \\
&\lim_{x\to 0^+} \frac{f_a(x)}{x^\alpha} = \infty \qquad \text{for every } \alpha > 0 \quad ,
\end{aligned}
\tag{3.2}
$$

hence, the theorems of Section 2 do not apply to $f_a(x)$. We can, however, establish the nonexistence of the first negative moment by noting

$$\int_0^{\delta} \frac{1}{x\log(1/x)}\,dx = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\delta} \frac{1}{x\log(1/x)}\,dx = \lim_{\epsilon\to 0^+} \{-\log[\log(1/x)]\}_{\epsilon}^{\delta}$$
$$= \lim_{\epsilon\to 0^+} \log\log(1/\epsilon) - \log[\log(1/\delta)] = \infty \quad . \tag{3.3}$$

In general, a condition involving only limits will not yield a necessary and sufficient condition for existence of moments; the condition must itself involve an integral. Feller [2, Sec. V.6] presents a theorem which gives a necessary and sufficient condition for the existence of the first positive moment. We present a similar theorem, which gives a necessary and sufficient condition for existence of the first negative moment.

THEOREM 3.1. _Let_ $f(x)$ _be a continuous density on_ $(0,\infty)$. $E(X^{-1})<\infty$ _if and only if_

$$\int_0^{\infty} \frac{1}{x^2} F(x)dx < \infty \quad , \tag{3.4}$$

where $F(x) = \int_0^x f(t)dt$ _is the distribution function associated with the density function_ f.

Proof. The proof follows from establishing the identity

$$\int_0^{\infty} \frac{1}{x} f(x)dx = \int_0^{\infty} \frac{1}{x^2} F(x)dx \quad , \tag{3.5}$$

in the sense that one side is finite if and only if the other side is finite.

For any $\delta > 0$, each integrand in (3.5) has a finite integral over $[\delta,\infty)$, so we can use integration by parts to establish

$$\int_{\delta}^{\infty} \frac{1}{x} f(x)dx = \frac{-F(\delta)}{\delta} + \int_{\delta}^{\infty} \frac{1}{x^2} F(x)dx \quad . \tag{3.6}$$

Now suppose the left-hand side of (3.5) is finite. This implies that $f(0) = 0$, and from L'Hospital's rule, we have

$$\lim_{\delta\to 0} \frac{F(\delta)}{\delta} = \lim_{\delta\to 0} f(\delta) = 0 \quad . \tag{3.7}$$

Thus, letting $\delta \to 0$ in (3.6) establishes (3.5) and the 'only if' part of the theorem.

Now, suppose that the right-hand side of (3.5) is finite. From (3.6), we have for every $\delta > 0$,

$$\int_{\delta}^{\infty} \frac{1}{x} f(x)dx \le \int_{\delta}^{\infty} \frac{1}{x^2} F(x)dx \quad . \tag{3.8}$$

Letting $\delta \to 0$ in (3.8) establishes $E(X^{-1}) < \infty$, and the proof is complete.

As mentioned before, the condition given in Theorem 3.1 is not nearly as easy to verify as the conditions in Section 2. In fact, in many cases, it is just as easy to check directly whether $E(X^{-1})$ exists, rather than to check the condition of Theorem 3.1. Fortunately, most common density functions are well enough behaved to be covered by the theorems in Section 2; only the more pathological ones, such as (3.1), will not be covered. But the density in (3.1) does serve a purpose: it illustrates that the question of existence of negative moments is a delicate one.

4. **Generalizations to Higher Order Moments.** The theorems presented in Sections 2 and 3 quickly generalize to cover higher order negative moments, i.e., $E(|X|^{-\beta})$, $\beta > 1$. The proofs are similar to those already presented, and will be omitted.

Analogous to the theorems of Section 2, we have the following two theorems:

THEOREM 4.1. *Let $f(x)$ be a continuous pdf on (a,b), $-\infty \le a < b \le \infty$. If one of the following conditions is true:*

(i) $0 \in (a,b)$ *and* $f(0) > 0$,

(ii) $a = 0$ *and* $\lim_{x \to 0^+} f(x) > 0$,

(iii) $b = 0$ *and* $\lim_{x \to 0^-} f(x) > 0$,

then $E(|X|^{-\beta}) = \infty$.

THEOREM 4.2. *Let* $f(x)$ *be a continuous pdf on* $(0,\infty)$. *If there exists* $\alpha > 0$ *such that*

$$\lim_{x \to 0^+} \frac{f(x)}{x^{\alpha+\beta-1}} < \infty \quad , \tag{4.1}$$

then $E(X^{-\beta}) < \infty$.

The integration-by-parts argument used in Theorem 3.1 remains valid for higher order moments. Thus, a proof similar to that of Theorem 3.1 can be used to establish the following theorem.

THEOREM 4.3. *Let* $f(x)$ *be a continuous density function on* $(0,\infty)$. $E(X^{-\beta}) < \infty$ *if and only if*

$$\int_0^\infty \left(F(x)/x^{\beta+1}\right) dx < \infty \quad , \tag{4.2}$$

where $F(x) = \int_0^x f(t)dt$.

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